WEAK-FIELD LIMIT OF THE CONFORMAL THEORY OF GRAVITY AND GALACTIC ROTATION CURVES

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ABSTRACT. We study the weak-field limit of the static spherically symmetric solution of the locally conformally invariant theory, which is regarded as an alternative to Einstein's general relativity theory in explaining the flat galactic rotation curves. In contrast with the previous works, we consider the physically relevant case where the scalar field that breaks conformal symmetry and generates fermion masses is nonzero. In the physical gauge, in which this scalar field is constant in space-time, the solution reproduces the weak-field limit of the Schwarzschild–(anti) de Sitter solution modified by an additional term that, depending on the sign of the Weyl term in the action, is either oscillatory or exponential as a function of the radial distance. Such behaviour reflects the presence of, correspondingly, either a tachyon or a massive ghost in the spectrum, which is a drawback of the theory under discussion.

Key words: Gravity: conformal; galaxies: rotation curves.

One of the long-standing problems of modern cosmology is the so-called problem of dark matter [see, e.g., Peebles (1993)]. In general, this problem consists in the discrepancy between the amount of the observed luminous matter on various spatial scales and the assumption that this matter is the only essential source of gravitation. Thus, on very large scales, the amount of the present luminous matter is insufficient to account for the measured rate of expansion of the universe. On the galactic scales, the problem reveals itself, in particular, in a peculiar behaviour of the galactic rotation curves in spiral galaxies, which do not appropriately fall with the distance from the galactic centre. In a broad sense, the problem can be stated as the violation of the laws of the general relativity theory and can be expressed in the form of the inequality

$$G_{\mu\nu} \neq 8\pi T^*_{\mu\nu}\,, \tag{1}$$

where $T^*_{\mu\nu}$ is the stress-energy tensor of the *observed* luminous matter and $G_{\mu\nu}$ is the Einstein tensor corresponding to the spacetime metric inferred from *observations* (we use the geometrized units, in which Newton's gravitational constant and the speed of light are equal to unity).

The common general solution of the above problem lies in the assumption that most of matter on the relevant spatial scales is invisible, so that an extra term should actually be present on the right-hand side of the above relation, thus restoring the equality. There exist several candidates for such dark matter, ranging from massive relic elementary particles (of known or predicted species) to compact objects of planetary type. The search for this dark-matter component is currently being continued.

In parallel to the above-mentioned common approach, some people consider another interesting possibility, namely, that it is not the right-hand side of Eq. (1) that is to be modified by the contribution from still undetected matter, but rather that it is its left-hand side that is to be somehow modified. In other words, it is assumed that the laws of the general relativity theory fail to be universally valid and must be replaced by some other laws. On this path, one certainly needs some guiding principles to decide how such a modification might be made.

Recently, Mannheim and Kazanas (1989) [see also Mannheim (1993, 1997, 1998), and references therein] explored the possibility that gravity is described by the conformally invariant theory with the key ingredient in the action being the Weyl term

$$\begin{aligned} I_{\mathrm{W}} &= -\alpha \int d^4x\sqrt{-g}\, C_{\lambda\mu\nu\kappa}C^{\lambda\mu\nu\kappa} \\ &= -2\alpha \int d^4x\sqrt{-g}\,\left(R_{\mu\nu}R^{\mu\nu} - R^2/3\right) \\ &\quad + \text{boundary terms}\,, \end{aligned} \tag{2}$$

where $C^{\lambda}{}_{\mu\nu\kappa}$ is the conformal Weyl tensor and α is the purely dimensionless gravitational coupling constant. In particular, they obtained the complete *conformally* static spherically symmetric solution of the theory described by Eq. (2) with the line element given by

$$ds^2 = C^2(x)\left[-G(r)dt^2 + dr^2/G(r) + r^2 d\Omega\right], \tag{3}$$

where $C(x)$ is an arbitrary nonzero function of the spacetime coordinates x, and $G(r)$ is given by

$$G(r) = 1 - \beta(2 - 3\beta\gamma)/r - 3\beta\gamma + \gamma r - \kappa r^2. \quad (4)$$

Here, β, γ, and κ are integration constants. Having tacitly assumed that test bodies move along the geodesics of the metric with the line element of Eq. (3) and with $C(x) \equiv 1$, Mannheim and Kazanas (1989) then claimed to recover the Newtonian term ($\propto 1/r$) in the potential of solution (4) of the conformal gravity theory and also suggested that the additional linear term γr in Eq. (4) might account for the flat galactic rotation curves without having to invoke dark matter.

It should be noted, however, that solution (3), (4) of the purely gravitational conformal theory defined by Eq. (2) is not quite relevant to the observations, since it is obtained without regard for the matter part of the theory that includes the mass-generation mechanism for the elementary particles and thereby for test bodies such as stars and planets. Such a feature of this solution is reflected in the unrestricted freedom of choosing the conformal factor $C(x)$ in Eq. (3) which clearly affects the timelike geodesics of the metric, but which is totally undetermined thus far. Moreover, the electrovac generalization of solution (3), (4) was previously obtained by Riegert (1984), who also asserted that one of the integration constants can be eliminated by further coordinate and conformal transformations. This property of the solution, with γ being such a constant, was noted and explicitly demonstrated by Schmidt (1984, 1999). All this makes very problematic the use of the metric given by the second expression in Eq. (3) and by Eq. (4) as an observable one.

Here, we consider this problem taking the matter to be represented by the generic conformally invariant action

$$\begin{aligned} I_{\rm M} &= -\int d^4x\sqrt{-g}\left[\partial^\mu S\partial_\mu S/2 + \lambda S^4\right. \\ &\quad \left. - S^2R/12 + i\bar{\psi}\gamma^\mu(x)\nabla_\mu\psi - \zeta S\bar{\psi}\psi\right], \quad (5) \end{aligned}$$

where ψ is the fermion field, S is the scalar field, R is the curvature scalar of the metric, and λ and ζ are dimensionless coupling constants. In the theory defined by Eqs. (2), (5), once the scalar field S is everywhere nonzero it can be gauged to an identical constant S_0 by a conformal transformation. In this gauge, the fermion part of the action acquires the standard form with constant mass, hence all physical effects receive the standard description; in particular, massive particles and test bodies move along the timelike geodesics of the metric as in the general relativity theory. It is clear that since conformal symmetry is broken and there are massive particles in the real world, one should take solutions with S being nonzero. The physical vacuum is then regarded as the state without excitations of the rest of the matter fields, in our case, the field ψ.

We consider solutions outside a compact source formed by the matter fields (represented in our model by the single field ψ). The equations of the theory have the form

$$4\alpha W_{\mu\nu} = T_{\mu\nu}, \quad (6)$$

where the two sides stem, respectively, from the variation of actions (2) and (5) with respect to the metric, and the expression of the stress-energy tensor $T_{\mu\nu}$ in the gauge $S \equiv S_0$ and with the ψ field being zero is given by

$$T_{\mu\nu} = -S_0^2\left(R_{\mu\nu} - g_{\mu\nu}R/2\right)/6 - \lambda S_0^4 g_{\mu\nu}. \quad (7)$$

Equations (6) with the right-hand side given by Eq. (7) are nothing but the Bach–Einstein equations with the cosmological constant term—the last term in Eq. (7). Note that the left-hand side of Eq. (6) is identically traceless, and it is convenient to rewrite system (6) as

$$4\alpha W_{\mu\nu} = \mathcal{T}_{\mu\nu}, \quad R = 24\lambda S_0^2, \quad (8)$$

where $\mathcal{T}_{\mu\nu} \equiv -S_0^2\left(R_{\mu\nu} - g_{\mu\nu}R/4\right)/6$ is the traceless part of the stress-energy tensor $T_{\mu\nu}$, and the second equation of system (8) is the trace of Eq. (6).

We restrict ourselves to the static spherically symmetric case. As we explained above, we are interested in the situation where $T_{\mu\nu}$ is given by Eq. (7) with constant nonzero S_0. In this gauge, a static spherically symmetric metric can be put in the form

$$ds^2 = -B(r)dt^2 + A(r)dr^2 + r^2 d\Omega\,. \quad (9)$$

It appears to be difficult to obtain the exact general solution for $A(r)$ and $B(r)$. However, it is possible to obtain solution in the weak-field limit. Let the physical metric of Eq. (9) in the spatial region of interest be sufficiently close to the flat one, so that

$$A(r) = 1 + \epsilon a(r), \quad B(r) = 1 - \epsilon b(r), \quad (10)$$

where ϵ is an auxiliary small parameter to be set equal to unity in the end.

To obtain the system of equations for the functions $a(r)$ and $b(r)$, we must linearize the equations of system (8) for the metric of in the small parameter ϵ. We note that the scalar curvature R of this metric is of order ϵ. Hence, the second equation of system (8) implies that the dimensionless value of $\lambda S_0^2 r^2$ should also be at least of order ϵ in the spatial region under consideration. On observational grounds, this restriction on the value of $\lambda S_0^2 r^2$ is quite natural since this value represents the effect of the cosmological constant, which is believed to be small on the galactic and stellar spatial scales. However, from the theoretical viewpoint, such a restriction constitutes the fine-tuning problem of the cosmological constant. The solution of this longstanding problem is absent, so we formally replace λ by $\epsilon\lambda$, thus taking into account the smallness of the corresponding parameter.

Omitting the calculations, which can be found in Barabash and Shtanov (1999), we present here the resulting solution. We make the notation

$$p = \frac{S_0^2}{24\alpha}, \quad q = \lambda S_0^2, \tag{11}$$

and note that solution depends on the sign of the constant p that coincides with the sign of α. First, we consider the case where $p > 0$. We obtain

$$a(r) = 2m/r - 2qr^2 + n\left[\sin(kr+\phi)/r - k\cos(kr+\phi)\right], \tag{12}$$

$$b(r) = \left[2m + 2n\sin(kr+\phi)\right]/r - 2qr^2, \tag{13}$$

where $k = \sqrt{p}$, and n and ϕ are integration constants. We see that in the Newtonian limit, apart from the universal term qr^2, there arises the additional gravitational potential

$$V(r) = -\frac{m + n\sin(kr+\phi)}{r}, \tag{14}$$

in which the constants m, n, and ϕ are to be related to the source. The constants $k = \sqrt{p}$ and q are universal and are given by Eq. (11).

We note that the linearized static spherically symmetric solutions in a generic (not conformally invariant) second-order gravitational theory without the cosmological constant were obtained by Stelle (1978). Their structure is similar to that of Eqs. (12), (13) and to solutions (25), (26) below. However, it is not possible to pass to the direct limit of conformal invariance in the solutions of Stelle (1978), because the case of conformal invariance is characterized by a nontrivial degeneracy, in particular, the massive scalar degree of freedom that is present in a generic case is missing here [see also Schmidt (1985*a*, 1985*b*, 1986) in this respect].

Now suppose that a static compact source is composed of identical "atoms" (these may be real atoms or elementary particles) and that each of these atoms produces static gravitational potential as given by Eq. (14) with identical constants m, n, and ϕ. In view of the weakness of the potential, we also assume the validity of the superposition principle. Then, if $\mu(\mathbf{r})$ is the spatial distribution of the "atoms" in the source, the total potential is given by the expression

$$\Phi(\mathbf{r}) = \int V(|\mathbf{r} - \mathbf{r}'|)\,\mu(\mathbf{r}')\,d\mathbf{r}'. \tag{15}$$

This potential is the sum of two terms: $\Phi(\mathbf{r}) = \Phi_m(\mathbf{r}) + \Phi_n(\mathbf{r})$. They satisfy the equations

$$\Delta\Phi_m(\mathbf{r}) = 4\pi m\mu(\mathbf{r}), \tag{16}$$

$$\Delta\Phi_n(\mathbf{r}) + p\,\Phi_n(\mathbf{r}) = 4\pi n\sin\phi\,\mu(\mathbf{r}), \tag{17}$$

that, in the theory under investigation, correspond to the unique Poisson equation of the linearized general relativity theory.

For a spherically symmetric compact distribution $\mu(r)$, the potential given by Eq. (15) with the kernel given by Eq. (14) is easily calculated:

$$\Phi(r) = -\int_r^\infty \frac{M(r')}{r'^2}dr' - \frac{N\sin(kr+\phi)}{r} - \frac{4\pi n\sin\phi}{kr}\int_r^\infty \mu(r')\sin\left[k(r-r')\right]r'dr', \tag{18}$$

where

$$M(r) = 4\pi m\int_0^r \mu(r')r'^2dr', \tag{19}$$

$$N = \frac{4\pi n}{k}\int_0^\infty \mu(r')\sin(kr')\,r'dr'. \tag{20}$$

Thus, outside the source, the potential of the form (14) is reproduced with the same phase ϕ, but with different coefficients m and n. Moreover, while the coefficient m is additive (it plays the role of the gravitational mass of the source), the coefficient n is not: its new value N is given by Eq. (20). However, the coefficient n becomes approximately additive for a distribution whose spatial size is significantly less than $1/k$.

If the product $kr < 1$ in the region of interest (say, on galactic scales), one can expand the oscillatory part of Eq. (14) in powers of kr to obtain

$$V(r) = V_0 - \frac{M_0}{r} + \frac{\Gamma r}{2} + Qr^2 + \mathcal{O}\left[(kr)^3\right], \tag{21}$$

where $V_0 = -nk\cos\phi$, $M_0 = m + n\sin\phi$, $\Gamma = nk^2\sin\phi$, and $Q = q + nk^3\cos\phi/6$. We thus recover the linear term in the potential of Eq. (21), similar to that which occurs in Eq. (4) and which was used by Mannheim and Kazanas (1989) to account for the flat galactic rotation curves. However, there exists an important observational bound that rules out the possibility for the linear term in expansion (21) to play a significant role on galactic scales. Note that the coefficients $-g_{00}(r)$ and $g_{rr}(r)$ of the metric of our solution are not mutually inverse, which is reflected in the fact that the functions $a(r)$ and $b(r)$, given, respectively, by Eqs. (12) and (13), are not equal to each other. At small enough distances, both functions reproduce the Newtonian potentials with the masses, respectively, $m_0 = m + n\sin\phi$ and $m_1 = m + (n\sin\phi)/2$, the difference between them being $\Delta m = (n\sin\phi)/2$. At the same time, the *Viking* spacecraft observations in the vicinity of the Sun indicate that the ratio $\Delta m/m \lesssim 2\times10^{-3}$ [see Will (1993)]. This implies the following observational bound for the Sun:

$$\frac{n\sin\phi}{m} \lesssim 4\times10^{-3}. \tag{22}$$

Since we assume that the parameter k is sufficiently small so that expansion (21) is legitimate on galactic scales, the values of both m and n are additive on such scales and estimate (22) is valid on galactic scales as well. Now, the linear term in Eq. (21) formally becomes comparable in magnitude to the Newtonian one only at the distance $r \sim \sqrt{M/\Gamma} \approx \sqrt{m/(nk^2 \sin\phi)}$. But, for such distances, we would have $kr \sim \sqrt{m/(n\sin\phi)} \gtrsim 10$ because of estimate (22), which contradicts the original assumption $kr < 1$. Thus, the linear term in expansion (21) cannot play a significant role on galactic scales, and one should rather try the exact potential in the form (18) for a spherically symmetric source with the bounding condition (22) to account for the galactic rotation curves.

It is instructive to estimate the realistic value of the constant α in Eq. (2) for which the value of kr is of order unity on a typical galactic scale of 10 kpc, thus making the potential of the form (14) in principle relevant to the galactic rotation curves. Whatever scalar fields are present in the theory, they all contribute to the value of p given by Eq. (11). Thus, at least the scalar Higgs field of the standard model of strong and electroweak interactions should be taken into account. The mean value of this field is known to be $\eta \simeq 246$ Gev; this value will contribute to S_0 in Eq. (11) and, in order that $k \times (10\text{ kpc}) \lesssim 1$ be valid, we must have

$$\alpha \gtrsim 10^{74}\,, \tag{23}$$

which, of course, is a severe restriction. It is difficult to conceive models in which this restriction is substantially weakened without fine-tuning.

On the other hand, if we take $\alpha \sim 1$, then the expectation value $\eta \simeq 246$ GeV of the standard model Higgs field leads to the spatial scale

$$1/k \sim 10^{-16}\text{ cm} \tag{24}$$

on which potential (14) oscillates. Its significance might only be manifest on the spatial scales of elementary particles, where, of course, the whole theory must be quantized.

In the case $p < 0$, which corresponds to $\alpha < 0$, the solution for $a(r)$ and $b(r)$ has the form

$$\begin{aligned} a(r) &= 2m/r - 2qr^2 \\ &+ n_1\,(1+kr)\,e^{-kr}/r + n_2\,(1-kr)\,e^{kr}/r, \end{aligned} \tag{25}$$

$$b(r) = 2m/r - 2qr^2 + 2n_1 e^{-kr}/r + 2n_2 e^{kr}/r, \tag{26}$$

where now $k = \sqrt{-p}$, and n_1 and n_2 are integration constants. Similar solutions in a generic second-order gravitational theory (not conformally invariant) without the cosmological constant were obtained by Stelle (1978). Solutions in the conformally invariant second-order theory with the Einstein term but without the cosmological-constant term were also obtained in Schmidt (1985*a*, 1985*b*, 1986). The physically meaningful solution is selected by imposing boundary conditions at infinity, which leads to the condition $n_2 = 0$. For sufficiently small values of k, the observational bound similar to condition (22) implies

$$\frac{n_1}{m} \lesssim 4 \times 10^{-3}, \tag{27}$$

and makes the extra exponential potential in Eq. (26) uninteresting.

Finally, we note that in the case $p < 0$, which corresponds to $\alpha < 0$, one also can obtain solutions by formally replacing the trigonometric functions in Eqs. (12), (13), and (14) by their hyperbolic counterparts and taking $k = \sqrt{-p}$. Equations (16), (17) will then remain valid in this case as well, with the replacement of $\sin\phi$ by $\sinh\phi$. The structure of the left-hand sides of Eqs. (16), (17) reflects, besides the presence of the massless graviton, also the well-known presence of a spin-two tachyon (in the case of $\alpha > 0$) or a spin-two massive ghost (in the case of $\alpha < 0$) on the background with $S \neq 0$ of the theory described by Eqs. (2), (5) [see, e.g., Stelle (1978)]. The presence of a tachyon in the case $\alpha > 0$ indicates instability of a large class of solutions, including the flat space-time solution in the case $\lambda = 0$; and the presence of a ghost in the case $\alpha < 0$ implies possible absence of perturbative unitarity in the corresponding quantum theory. This appears to be the main drawback of the conformal theory under discussion.

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References

Barabash O.V., Shtanov Yu.V.: 1999, *Phys. Rev. D*, **60**, 064008.
Mannheim P.D., Kazanas D.: 1989, *Ap. J.*, **342**, 635.
Mannheim P.D.: 1993, *Ap. J.*, **419**, 150.
Mannheim P.D.: 1997, *Ap. J.*, **479**, 659.
Mannheim P.D.: 1998, *Phys. Rev. D*, **58**, 103511.
Peebles P.J.E.: 1993, *Principles of Physical Cosmology* (Princeton University Press, Princeton, N.J.).
Riegert R.J.: 1984, *Phys. Rev. Lett.*, **53**, 315.
Schmidt H.-J.: 1984, *Ann. Phys. (Leipz.)*, **41**, 435.
Schmidt H.-J.: 1985*a*, *Astron. Nachr.*, **306**, 67.
Schmidt H.-J.: 1985*b*, *Astron. Nachr.*, **306**, 231.
Schmidt H.-J.: 1986, *Astron. Nachr.*, **307**, 339.
Schmidt H.-J.: 1999, gr-qc/9905103, Report No. UNIPO-MATH-99-May-27.
Stelle K.S.: 1978, *Gen. Rel. and Grav.*, **9**, 353.
Will C.M.: 1993, *Theory and Experiment in Gravitational Physics* (Cambridge University Press, Cambridge).