PHYSICAL PROPERTIES OF A CLASS OF SPHERICALLY SYMMETRIC PERFECT FLUID DISTRIBUTIONS IN NONCOMOVING COORDINATES

H. Knutsen

Stavanger College, P.O. Box 2557, Ullandhaug, N-4091 Stavanger, Norway

ABSTRACT. The physical properties of an exact solution of Einstein's field equations are examined. This spherically symmetric perfect fluid solution contains expansion, acceleration and shear. There exist models with regions of spacetime where the pressure and the density are positive and the dominant energy condition and the causality condition are also fulfilled. Moreover, the pressure and the density gradients are equal. The mass function is zero at the origin where there is Lorentz-Minkowski geometry and no trapped surface exists.

Key words: Noncomoving coordinates; expansion; acceleration; shear.

1. Introduction and line element

An exact solution of Einstein's field equations which appear in simple form in non-comoving coordinates may show a bewildering appearance when transformed to a comoving system. Hence, McVittie and Wiltshire (1977) thought it worth while to examine the possibility of solving Einstein's equations in terms of noncomoving coordinates. However, they obtained their solutions by ancillary mathematical assumptions and abstained from a detailed physical analysis of their models. In this paper we examine a particular model given by McVittie and Wiltshire (1977). We shall examine the properties of the following line element

$$\begin{aligned} ds^2 &= P^{4/3}d\eta^2 - P^{2/3}e^{-2\varepsilon\eta/\eta_0}\left(\sin\frac{\xi}{2}\right)^{-4} \\ &\times \quad (d\xi^2 + \sin^2\xi d\Omega^2), \end{aligned} \tag{1}$$

where $P = 1 + Ae^{-2z}$, $e^z = \sin\frac{\xi}{2}e^{\varepsilon\eta/\eta_0}$, $\varepsilon = \pm 1$ and A is a constant which may be positive or negative and η_0 is one more constant. This metric is solution (8.11) in McVittie and Wiltshire (1977). Writing

$$R^2 = P^{2/3}e^{-2\varepsilon\eta/\eta_0}\left(\sin\frac{\xi}{2}\right)^{-4}\sin^2\xi, \tag{2}$$

metric (1) yields that the centre (R=0) is represented by $\xi = \pi$. Using Einstein's equations "in reverse", we now find after some calculations that the pressure p and the density ρ are respectively given by

$$\begin{aligned} 8\pi p &= \frac{28 - 20P - 35P^2}{9\eta_0^2 P^{10/3}} + e^{2\varepsilon\eta/\eta_0}\frac{(P-1)\sin^2\frac{\xi}{2}}{9P^{8/3}} \\ &\times \quad \left[4P + 5 + 5(P-1)\sin^2\frac{\xi}{2}\right], \end{aligned} \tag{3}$$

$$\begin{aligned} 8\pi\rho &= \frac{3(5P-2)^2}{9\eta_0^2 P^{10/3}} \\ &- e^{2\varepsilon\eta/\eta_0}\frac{(P-1)\sin^2\frac{\xi}{2}}{9P^{8/3}}\left[4P + 5 + 5(P-1)\sin^2\frac{\xi}{2}\right] \\ &- \frac{5}{18}e^{2\varepsilon\eta/\eta_0}\frac{(P-1)^2}{P^{8/3}}\sin^2\xi. \end{aligned} \tag{4}$$

For the centre ($\xi = \pi$) we have

$$8\pi p_c = \frac{28 - 20P_c - 35P_c^2}{9\eta_0^2 P_c^{10/3}} + e^{2\varepsilon\eta/\eta_0}\frac{P_c - 1}{P_c^{5/3}}, \tag{5}$$

$$8\pi\rho_c = \frac{3(5P_c - 2)^2}{9\eta_0^2 P_c^{10/3}} - e^{2\varepsilon\eta/\eta_0}\frac{P_c - 1}{P_c^{5/3}}, \tag{6}$$

where the suffix c denotes centre value. We now choose $P_c = 1 + Ae^{-2\varepsilon\eta/\eta_0}$ such that the following two conditions are fulfilled:

$$28 - 20P_c - 35P_c^2 > 0, \tag{7}$$

$$3(5P_c - 2)^2 > 28 - 20P_c - 35P_c^2. \tag{8}$$

It is easily seen that this will be the case if and only if

$$P_c \in \langle -1.225\ ,\ -0.241\rangle \cup \langle\ 0.604\ ,\ 0.653\ \rangle. \tag{9}$$

Next, let $e^{2\varepsilon\eta/\eta_0} \approx 0$ such that the first two terms of equations (5) and (6) will dominate the expressions for the pressure and the density. Hence, we have the important conclusion *there are classes of solutions for which there exist regions of spacetime in which these models are physically valid.* Moreover, these models are nonsingular close to the centre. From equations (3) and (4) it is further seen that the weak energy condition $\rho + p \geq 0$ yields

$$H \equiv \frac{16}{\eta_0^2}e^{-2\varepsilon\eta/\eta_0} - P^{2/3}\sin^2\xi \geq 0. \tag{10}$$

2. The four velocity and comoving coordinates

We choose the timelike component u^4 to be positive, and with our line element (1) we obtain

$$u^4 = \frac{4}{\eta_0} P^{-2/3} e^{-\varepsilon\eta/\eta_0} H^{-1/2}. \tag{11}$$

$$u^1 = -\varepsilon \sin\xi \sin^2\frac{\xi}{2} e^{\varepsilon\eta/\eta_0} H^{-1/2}. \tag{12}$$

We shall now examine if it is really possible to transform our metric (1) into comoving coordinates. The condition expressing orthogonality of the metric and the condition that the radial coordinate r is comoving yield the following two differential equations for the time coordinate t and for r

$$e^{2\mu} u^1 \frac{\partial t}{\partial\eta} + e^{2\lambda} u^4 \frac{\partial t}{\partial\xi} = 0, \tag{13}$$

$$u^1 \frac{\partial r}{\partial\xi} + u^4 \frac{\partial r}{\partial\eta} = 0. \tag{14}$$

Equation (13) simplifies beautifully and we obtain

$$t = e^{\varepsilon\eta/\eta_0} \sin\frac{\xi}{2}. \tag{15}$$

However, with the substitutions $x = e^{-2\varepsilon\eta/\eta_0}$ and $y = \sin\frac{\xi}{2}$ we find that equation (14) reads

$$(ay^2 + bx)^{2/3}(1-y^2)y^{5/3}\frac{\partial r}{\partial y} + x^2\frac{\partial r}{\partial x} = 0, \tag{16}$$

where a and b are arbitrary nonzero constants. We have not been able to integrate equation (16). Hence, we can not write the solution in comoving coordinates. But we shall still be able to discuss several interesting physical aspects concerning this model.

3. Four velocity field

The expansion Θ is given by

$$\begin{aligned}
\Theta &= \varepsilon H^{-3/2}\{-\frac{64}{P^{5/3}}(5P-2)\frac{e^{-3\varepsilon\eta/\eta_0}}{\eta_0^4} \\
&+ \frac{8}{3}\left[(58 - 40\sin^2\frac{\xi}{2})\sin^2\frac{\xi}{2} - \frac{7}{P}\sin^2\xi\right]\frac{e^{-\varepsilon\eta/\eta_0}}{\eta_0^2} \\
&+ 2\sin^2\xi\left[\frac{1}{3P^{1/3}}\sin^2\xi\right. \\
&- \left.\frac{P^{2/3}}{3}\sin^2\frac{\xi}{2}(7 - 4\sin^2\frac{\xi}{2})\right]e^{\varepsilon\eta/\eta_0}\}.
\end{aligned} \tag{17}$$

The four-acceleration $\dot{u}_i$ reads

$$\dot{u}_1 = \frac{16}{\eta_0^2} e^{-2\varepsilon\eta/\eta_0} P^{2/3} \sin\xi \sin^2\frac{\xi}{2} H^{-2}, \tag{18}$$

$$\dot{u}_4 = \frac{4}{\eta_0}\varepsilon P^{4/3}\sin^2\xi\sin^4\frac{\xi}{2}H^{-2}. \tag{19}$$

The shear tensor σ_{ij} reads

$$\sigma_{11} = \frac{32\varepsilon}{9\eta_0^2} e^{-3\varepsilon\eta/\eta_0} P^{2/3} \frac{1}{\sin^2\frac{\xi}{2}} H^{-5/2} K, \tag{20}$$

$$\sigma_{14} = \frac{8}{9\eta_0} e^{-\varepsilon\eta/\eta_0} P^{4/3} \sin\xi H^{-5/2} K, \tag{21}$$

$$\sigma_{22} = -\frac{\varepsilon}{9} e^{-\varepsilon\eta/\eta_0} P^{2/3} \frac{\sin^2\xi}{\sin^2\frac{\xi}{2}} H^{-3/2} K, \tag{22}$$

$$\sigma_{44} = \frac{2\varepsilon}{9} e^{\varepsilon\eta/\eta_0} P^2 \sin^2\xi \sin^2\frac{\xi}{2} H^{-5/2} K, \tag{23}$$

where K is given by

$$\begin{aligned}
K &= \frac{16}{\eta_0^2}\left(1 - \sin^2\frac{\xi}{2}\right)\left(1 + \frac{2}{P}\right)e^{-2\varepsilon\eta/\eta_0} \\
&- \left[\left(1 + 2\sin^2\frac{\xi}{2}\right)P^{2/3} + \frac{2}{P^{1/3}}\left(1 - \sin^2\frac{\xi}{2}\right)\right]\sin^2\xi.
\end{aligned} \tag{24}$$

The shear invariant reads

$$\sigma_{ij}\sigma^{ij} = \frac{2}{27} e^{2\varepsilon\eta/\eta_0} \sin^4\frac{\xi}{2} H^{-3} K^2. \tag{25}$$

In Kramer et al. (1980) we find the following statement concerning the solutions given by McVittie and Wiltshire (1977), "not all of their solutions have nonzero shear!". With our new and previous results we can sharpen that statement and declare *the McVittie-Wiltshire solutions which are non static and not trivially conformally flat all have expansion, acceleration and shear.*

4. Sound speed and gradients

The speed of sound v_{sound} is given by

$$\begin{aligned}
v^2_{sound} &= 2\{-e^{4\varepsilon\eta/\eta_0}P^{4/3}\sin^2\xi\sin^2\frac{\xi}{2} \\
&\times [P + 2 + 2(P-1)\sin^2\frac{\xi}{2}] \\
&+ 4\frac{e^{2\varepsilon\eta/\eta_0}}{\eta_0^2}P^{2/3}\sin^2\frac{\xi}{2} \\
&\times [-2P\sin^2\frac{\xi}{2} + 11P + 22(1 - \sin^2\frac{\xi}{2})] \\
&- \frac{112}{\eta_0^4}(P+2)\} \times \{e^{4\varepsilon\eta/\eta_0}P^{4/3}\sin^2\xi\sin^2\frac{\xi}{2} \\
&\times [7P - 4 - 4(P-1)\sin^2\frac{\xi}{2}] \\
&+ 8\frac{e^{2\varepsilon\eta/\eta_0}}{\eta_0^2}P^{2/3}\sin^2\frac{\xi}{2} \\
&\times [20P\sin^2\frac{\xi}{2} - 29P + 14(1 - \sin^2\frac{\xi}{2})] \\
&+ \frac{96}{\eta_0^4}(5P-2)\}^{-1}.
\end{aligned} \tag{26}$$

The sound speed at the centre ($\xi = \pi$) reads

$$\begin{aligned} v^2_{sound}(centre) &= \left[28(P_c+2) - 9\eta_0^2 e^{2\varepsilon\eta/\eta_0} P_c^{5/3}\right] \\ &\times \left[-12(5P_c-2) + 9\eta_0^2 e^{2\varepsilon\eta/\eta_0} P_c^{5/3}\right]^{-1}. \end{aligned} \quad (27)$$

We now follow the process we used to obtain physically valid regions of spacetime, *i.e.* we restrict spacetime to regions where $e^{2\varepsilon\eta/\eta_0} \approx 0$. For these regions we have

$$v^2_{sound}(centre) = -\frac{7(P_c+2)}{3(5P_c-2)}. \quad (28)$$

We demand the sound speed to be real and less than the speed of light in vacuum. The following conditions must then be fulfilled

$$0 < -\frac{7(P_c+2)}{3(5P_c-2)} < 1, \quad (29)$$

Remembering condition (9) we obtain the following restriction

$$P_c \in \langle -1.225\ ,\ -0.364 \rangle. \quad (30)$$

The ratio of the gradients with respect to comoving radial coordinate r is given by

$$\begin{aligned} \left(\frac{\partial p}{\partial r}\right)\left(\frac{\partial \rho}{\partial r}\right)^{-1} &= \left(e^{2\lambda}u^4\frac{\partial p}{\partial \xi} + e^{2\mu}u^1\frac{\partial p}{\partial \eta}\right) \\ &\times \left(e^{2\lambda}u^4\frac{\partial \rho}{\partial \xi} + e^{2\mu}u^1\frac{\partial \rho}{\partial \eta}\right)^{-1} \end{aligned} \quad (31)$$

This expression simplifies beautifully and we find

$$\frac{\partial p}{\partial r} = \frac{\partial \rho}{\partial r}. \quad (32)$$

We thus have the remarkable fact that *the pressure gradient and the density gradient with respect to comoving radial coordinate are the same.*

5. Mass function

The mass function is given by

$$m = \frac{4}{3}\pi\rho R^3 + E, \quad (33)$$

where E is interpreted as pure gravitational field energy (not binding energy) within spheres of surface radius R. However, we also have

$$m = \frac{R}{2}\left[1 + g^{ij}(\frac{\partial R}{\partial x^i})(\frac{\partial R}{\partial x^j})\right]. \quad (34)$$

and we obtain

$$\begin{aligned} m &= \frac{4e^{-3\varepsilon\eta/\eta_0}(5P-2)^2}{9\eta_0^2 P^{7/3}}\cot^3\frac{\xi}{2} \\ &- \frac{4e^{-\varepsilon\eta/\eta_0}(4P-1)(P-1)}{9P^{5/3}}\cot\frac{\xi}{2} \\ &+ \frac{4e^{-\varepsilon\eta/\eta_0}(5P-2)(P-1)}{9P^{5/3}}\cot\frac{\xi}{2}\sin^2\frac{\xi}{2} \\ &- \frac{4e^{-\varepsilon\eta/\eta_0}(P-1)^2}{9P^{5/3}}\cot\frac{\xi}{2}\sin^4\frac{\xi}{2}. \end{aligned} \quad (35)$$

We further find

$$\begin{aligned} \frac{4}{3}\pi\rho R^3 &= \frac{4e^{-3\varepsilon\eta/\eta_0}(5P-2)^2}{9\eta_0^2 P^{7/3}}\cot^3\frac{\xi}{2} \\ &- \frac{4e^{-\varepsilon\eta/\eta_0}(14P-5)(P-1)}{27P^{5/3}}\cot\frac{\xi}{2} \\ &+ \frac{4e^{-\varepsilon\eta/\eta_0}(19P-10)(P-1)}{27P^{5/3}}\cot\frac{\xi}{2}\sin^2\frac{\xi}{2} \\ &- \frac{20e^{-\varepsilon\eta/\eta_0}(P-1)^2}{27P^{5/3}}\cot\frac{\xi}{2}\sin^4\frac{\xi}{2}. \end{aligned} \quad (36)$$

The gravitational field energy E, however, takes the simple form

$$E = \frac{8e^{-\varepsilon\eta/\eta_0}(P-1)^2}{27P^{5/3}}\cot\frac{\xi}{2}(1-\sin^2\frac{\xi}{2})^2, \quad (37)$$

and both the mass function m and the gravitational field energy E vanish at the centre of the matter distribution. We further obtain

$$\left(\frac{2m}{R}\right)_c = 0 \quad (38)$$

and we conclude that no apparent horizon or trapped surface exist close to the centre. The criterion to have Lorentz-Minkowski geometry at the origin is given by

$$B_c^2 = \left(\frac{\partial R}{\partial r}\right)_c^2. \quad (39)$$

We find

$$\left(\frac{\partial R}{\partial r}\right)^2 B^{-2} = \frac{16}{\eta_0^2}e^{-2\varepsilon\eta/\eta_0}\sin^4\frac{\xi}{2}H^{-1}. \quad (40)$$

Remembering definition (10) we find that *there is Lorentz-Minkowski geometry at the origin.*

References

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