HIGHLY RELATIVISTIC CIRCULAR ORBITS OF SPINNING PARTICLE IN THE SCHWARZSCHILD AND KERR FIELDS

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ABSTRACT. The solutions of the Mathisson-Papapetrou equations which describe highly relativistic circular orbits of a spinning particle in Schwarzschild's and Kerr's fields are considered. The domain of existence of those orbits and the necessary values of the particle's velocity for their realization are studied. These results can be applied while analyzing the synchrotron radiation in some astrophysical processes.

Key words: Mathisson-Papapetrou equations; highly relativistic spinning particle; gravitational field.

There are two possibilities to investigate the effects of interaction of the particle spin with the gravitational field on its motion in this field: (1) For a classical (nonquantum) particle the equations firstly derived by M.Mathisson [1] are effectively used; (2) For a quantum fermion particle one can use the general relativistic Dirac equation. It is important that in some quasi-classical limit the Mathisson equations follow from the Dirac equation. Just the Mathisson equations (also known as the Mathisson-Papapetrou equations) are in the focus of our consideration and now we continue the program of studying the specific properties of the highly relativistic motions of a spinning particle relative to the Schwarzschild and Kerr black holes. In this paper we present the results on highly relativistic essentially nongeodesic circular orbits of a spinning particle in the Schwarzschild and Kerr backgrounds. Note that practically any textbook on general relativity contains information concerning possible geodesic circular orbits of a spinless test particle in a Schwarzschild background as an important point of description of the black hole properties. On the contrary, the similar information on possible circular orbits of a spinning test particle in this background is not presented in the literature. Among other types of motions the circular highly relativistic orbits are of importance for investigations of possible synchrotron radiation, both electromagnetic and gravitational, of protons and electrons in the gravitational field of a black hole.

As in [2], we shall consider the Mathisson-Papapetrou equations under the Mathisson-Pirani supplementary condition in the Kerr metric using the Boyer-Lindquist coordinates $x^1 = r, \quad x^2 = \theta, \quad x^3 = \varphi, \quad x^4 = t$. However, in contrast to [2], where these equations were considered in some approximation, here we deal with the Mathisson-Papapetrou equations in their exact form. Our purpose is to investigate the equatorial circular orbits of a spinning particle in the plane $\theta = \pi/2$ of the Kerr source, when the spin is orthogonal to this plane. Then it follows from the MP equations that the space region of existence of the circular orbits and the dependence of the particle velocity for these orbits on their radius are determined by the solutions of set of the two algebraic equations. These equations can be written as

$$\begin{gathered}(\alpha^2 - y_1^3)y_7^2 - 2\alpha y_7 y_8 + y_8^2 - 3\alpha\varepsilon_0(y_1^2+\alpha^2)y_7^2 y_1^{-2}\\ +3\varepsilon_0(y_1^2+2\alpha^2)y_7 y_8 y_1^{-2} - 3\alpha\varepsilon_0 y_8^2 y_1^{-2}\\ +\alpha\varepsilon_0(3y_1^2+\alpha^2)(y_1^3-\alpha^2)y_7^4 y_1^{-3} - \alpha\varepsilon_0\left(1-\frac{2}{y_1}\right)y_8^4 y_1^{-3}\\ +\varepsilon_0(y_1^6 - 3y_1^5 - 3\alpha^2 y_1^3 + 9\alpha^2 y_1^2 + 4\alpha^4)y_7^3 y_8 y_1^{-3}\\ +\alpha\varepsilon_0(3y_1^3 - 11y_1^2 - 6\alpha^2 + 2\alpha^2 y_1^{-1})y_7^2 y_8^2\\ +\varepsilon_0(4\alpha^2 - 4\alpha^2 y_1^{-1} - y_1^3 + 3y_1^2)y_7 y_8^3 y_1^{-3} = 0, \end{gathered} \tag{1}$$

$$\begin{gathered}-\left(y_1^2+\alpha^2+\frac{2\alpha^2}{y_1}\right)y_7^2 + \frac{4\alpha y_7 y_8}{y_1}\\ +\left(1-\frac{2}{y_1}\right)y_8^2 = 1,\end{gathered} \tag{2}$$

where y_i are the dimensionless quantities connected with the particle coordinate and 4-velocity by the definition

$$y_1 = \frac{r}{M}, \quad y_2 = \theta, \quad y_3 = \varphi, \quad y_4 = \frac{t}{M},$$

$$y_5 = u^1, \quad y_6 = Mu^2, \quad y_7 = Mu^3, \quad y_8 = u^4,$$

here M is Kerr's mass. The values ε_0 and α in Eqs. (1) and (2) are determined as

$$\varepsilon_0 \equiv \frac{S_0}{mM}, \quad \alpha \equiv \frac{a}{M},$$

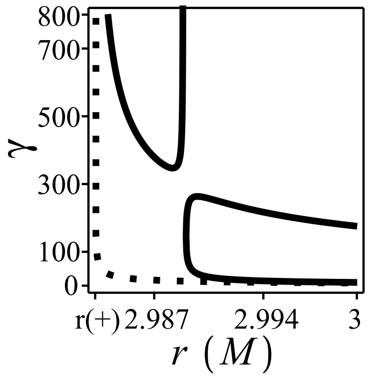

Figure 1: Dependence of the Lorentz factor on r for the highly relativistic circular orbits with $d\varphi/ds > 0$ of the spinning particle in Kerr's background at $a = 0.0145M$, $\varepsilon_0 = 0.01$ (solid lines). The dotted line corresponds to the geodesic circular orbits.

where $|S_0|$ is the absolute value of the particle spin and a is the Kerr angular momentum; according to the test condition for a spinning particle it is necessary $|\varepsilon_0| \ll 1$. Note that among eight values y_i only three of them, namely y_1, y_7 and y_8, are present in Eqs. (1) and (2), because for the circular motions we have $y_5 = 0$ and $y_6 = 0$. Therefore, for any fixed value of the radial coordinate, i.e. y_1, we have the two algebraic Eqs. (1) and (2) which let us find the necessary values of y_7 and y_8. By these values of y_1, y_7, y_8 one can calculate the relativistic Lorentz γ-factor of a moving particle as estimated by an observer which is at rest relative to the Kerr mass. The expression for this γ-factor for any circular motions in the equatorial plane is

$$\gamma = \left(1 - \frac{2}{y_1}\right)^{1/2} \left(y_8 + \frac{2\alpha y_7}{y_1 - 2}\right). \tag{3}$$

It is known that the geodesic equations in Kerr's background admit the highly relativistic circular orbits of a particle with the nonzero mass only in the small neighborhood of the values $r_{ph}^{(+)}$ and $r_{ph}^{(-)}$ that are the radial coordinates of the co-rotating and counter-rotating circular photon orbits.

Figures 1–4 illustrate both the domain of existence of the circular orbits and the dependence of the γ-factor on the radial coordinate for these orbits at some typical cases of different values of the Kerr parameter a, when $|\varepsilon_0| = 10^{-2}$. Figure 1 describe the case $a = 0.0145M$, when $r_{ph}^{(+)} \approx 2.983$ and $r_{ph}^{(-)} \approx 3.017$. Note that for $r \leq r_{ph}^{(+)}$ there are not any circular orbits of the spinning particle, i.e. this situation is the same as for the spinless particle. In the narrow space region between $r = r_{ph}^{(+)}$ and $r \approx 3.006M$ there are the highly relativistic circular orbits with the much higher Lorentz factor than it is necessary for the spinless particle (for comparison the dotted line in Fig. 1 shows the curve for the

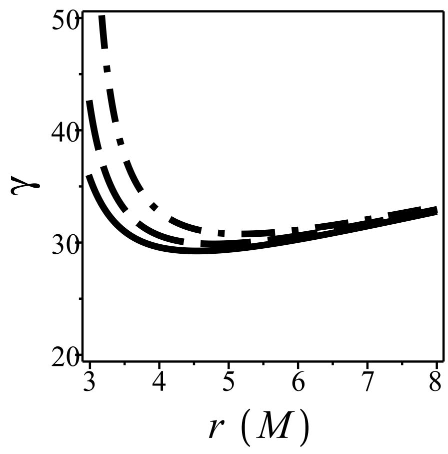

Figure 2: Lorentz factor vs. r for $\varepsilon_0 = 0.01$, $d\varphi/ds > 0$ at $a = 0.1M$ (dash-dotted line), $a = 0.5M$ (dashed line), and $a = M$ (solid line).

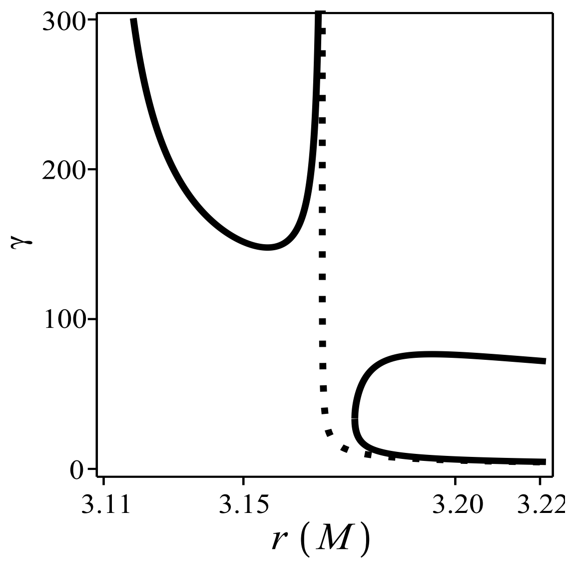

Figure 3: Lorentz factor vs. r for the highly relativistic circular orbits of the spinning particle in Kerr's background with $\varepsilon_0 = -0.01$, $d\varphi/ds < 0$ of the spinning particle in Kerr's background at $a = 0.15M$. The dotted line corresponds to the geodesic circular orbits with $r_{ph}^{(-)} \approx 3.17M$.

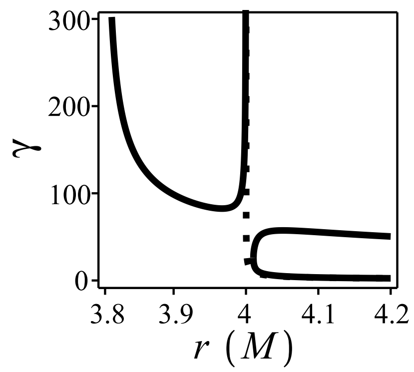

Figure 4: Lorentz factor vs. r for the highly relativistic circular orbits of the spinning particle in Kerr's background with $\varepsilon_0 = -0.01$, $d\varphi/ds < 0$ of the spinning particle in Kerr's background at $a = M$. The dotted line corresponds to the geodesic circular orbits with $r_{ph}^{(-)} = 4M$.

geodesic motion). We stress that the existence of those orbits is caused by the interaction of the particle's spin with the angular momentum of Kerr's source: in the Schwarzschild background the corresponding orbits are absent according to Fig. 1 from [3]. Also note that by the solutions of Eqs. (1) and (2) the necessary value γ tends to ∞ if a tends to 0. In the wide space region for r larger than $r \approx 3.006M$ Eqs. (1) and (2) admits for any fixed value r the two circular orbits and in this sense Fig. 1 below is similar to Fig. 1 from [3] for the Schwarzschild background.

Figure 2 shows the dependence γ on r for different values a in the space domain far from $r = r_{ph}^{(+)}$. For r grater than $r \approx 8M$ all the curves in Fig. 2 tend to the corresponding curve for Schwarzschild's case. We stress that all the curves $\gamma(r)$ for the spinning particle lay above the corresponding geodesic lines, i.e. here the spin-gravity interaction causes an additional significant attractive action as compare to the usual geodesic attraction.

In contrast to Figs. 1 and 2, Figs. 3 and 4 show the situations with another direction of the particle orbital motion, when $d\varphi/ds < 0$, and for the opposite orientation of the spin, when $\varepsilon_0 < 0$ (the last means that $S_\theta < 0$). The graphs in Figs. 3 and 4 which are located in the region with $r < r_{ph}^{(-)}$ describe the specific effects just in Kerr's field, in contrast to the graphs in the region $r > r_{ph}^{(-)}$ which are similar to the corresponding graphs for Schwarzschild's field. The existence of the orbits in the region $r < r_{ph}^{(-)}$ points out that in the case when $a > 0$, $\varepsilon_0 < 0$ and $d\varphi/ds < 0$ the spin-gravity interaction supplies the repulsive action on the spinning particle as compare to the motion of a spinless particle.

Since, according to Figs. 1–4 and others, which are not presented here for brevity, the highly relativistic circular orbits exist in Schwarzschild's and Kerr's fields in much wider space domain as compare to the case of a spinless particle.

Concerning the dependence of the Lorentz factor on the values of ε_0 we note that, as it is pointed out in [2], γ is proportional to $1/\sqrt{|\varepsilon_0|}$. The numerical estimates show that, for example, if M is equal to three of the Sun's mass (as for a black hole), then $|\varepsilon_0|$ for an electron is of order 4×10^{-17} and the sufficient value of γ-factor for the realization of some circular orbits is of order 2×10^8 [2]. For a neutrino with the mass $\approx 1eV$ we have $\varepsilon_0 = 2 \times 10^{-11}$ and $\gamma = 3 \times 10^5$. It is known that some particles from the cosmic rays posses these values.

References

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