THE INVARIANT SPLITTING FORMALISM IN GENERAL RELATIVITY

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ABSTRACT. The invariant splitting method of a pseudo-Riemanian manifold on arbitrary number of sections has been constructed in the present work.

Key words: general relativity, splitting, tangent bundle, the Gauss-Codacci-Ricci's equations.

Our technique is based on introduction of projection operators $\{H^\alpha\}$, inducing splitting of the bundle $T(M)$ and geometrical objects defined on it. The linear operators $\{H^\alpha\}$ (the tensors of (1,1) type) are given by:

$$\begin{aligned} H^\alpha \cdot H^\beta &= \delta^{\alpha\beta} \\ \sum_\alpha H^\alpha &= I \end{aligned} \tag{1}$$

where $\alpha,\ \beta = 1, 2, ...k$ are indeces of sections and I is the unit operator. Any vector $\vec{X}$ and one-form ω are projected into corresponding sections of the tangent and cotangent bundles by these projection operators $\{H^\alpha\}$

$$\begin{gathered} H^\alpha \cdot \vec{X}^\alpha \equiv H^\alpha(\vec{X}) = \vec{X}^\alpha, \vec{X}^\alpha \in \Sigma^\alpha \\ H^\alpha \cdot \omega \equiv H^\alpha(\omega) = \omega^\alpha, \omega^\alpha \in \Sigma^{*\alpha} \\ T(M) = \Sigma^1 \oplus \Sigma^2 \oplus ... \oplus \Sigma^k \\ \dim T(M) = \sum_{i=1}^k \dim \Sigma^i, \\ \vec{X} = \sum_{\alpha=1}^k \vec{X}^\alpha, \\ \Sigma^\alpha \equiv Im H^\alpha \end{gathered} \tag{2}$$

so that for any metric g we have

$$g = \sum_{\alpha=1}^k g^\alpha \tag{3}$$

where g^α is the metric induced on the section Σ^α. Due to conditions $H^\alpha \cdot H^\beta = 0$ and $g(H^\alpha \cdot \vec{X}, \vec{Y}) = = g(\vec{X} \cdot H^\alpha, \vec{Y})$ it follows that

$$\vec{X}^\alpha \cdot \vec{Y}^\beta = 0, g^\alpha(\vec{X}^\beta, \cdot) = 0 \tag{4}$$

The Levi-Civita connection ∇ is splitted on k^3 components

$$\nabla_{\vec{X}} \vec{Y} = \sum_{\alpha,\beta,\gamma} \nabla^\gamma_{\vec{X}^\alpha} \vec{Y}^\beta, \quad \alpha, \beta, \gamma = 1, 2, ...k \tag{5}$$

and the components

$$\nabla^\alpha_{\vec{X}^\beta} \vec{Y}^\beta = H^\alpha(\nabla_{\vec{X}^\beta} \vec{Y}^\beta) \equiv -B(\vec{X}^\beta, \vec{Y}^\beta)$$

determine the exterior unholonomisity tensor of the section Σ^β, while the components

$$\nabla^\alpha_{\vec{X}^\beta} \vec{Y}^\beta = H^\alpha(\nabla_{\vec{X}^\beta} \vec{Y}^\gamma) \equiv -Q^\alpha(\vec{X}^\beta, \vec{Y}^\gamma)$$

determine the generalized Ricci's coefficients of rotation.

Substituting all the connections in the invariant definition of the curvature tensor

$$R(\vec{X}, \vec{Y})\vec{Z} \cdot \vec{V} = (\nabla_{\vec{X}} \nabla_{\vec{Y}} - \nabla_{\vec{Y}} \nabla_{\vec{X}} - \nabla_{[\vec{X},\vec{Y}]})\vec{Z} \cdot \vec{V} \tag{6}$$

by their "splitted representatives" (??), we have obtained the invariant generalizations of the Gauss-Codacci-Ricci's equations:

$$\begin{gathered} R(\vec{X}^\alpha, \vec{Y}^\alpha)\vec{Z}^\alpha \cdot \vec{V}^\alpha = R^\alpha(\vec{X}^\alpha, \vec{Y}^\alpha)\vec{Z}^\alpha \cdot \vec{V}^\alpha + \\ + \sum_{\gamma \neq \alpha} \{2A^\gamma(\vec{X}^\alpha, \vec{Y}^\alpha) \cdot B^\gamma(\vec{Z}^\alpha, \vec{V}^\alpha) + \\ + B^\gamma(\vec{Y}^\alpha, \vec{V}^\alpha) \cdot B^\gamma(\vec{X}^\alpha, \vec{Z}^\alpha) - \\ - B^\gamma(\vec{X}^\alpha, \vec{Y}^\alpha) \cdot B^\gamma(\vec{Y}^\alpha, \vec{Z}^\alpha)\}, \end{gathered} \tag{7}$$

where $R^\alpha(\vec{X}^\alpha, \vec{Y}^\alpha)\vec{Z}^\alpha \cdot \vec{V}^\alpha$ is the curvature tensor on the section Σ^α.

$$\begin{aligned} R(\vec{X}^\alpha, \vec{Y}^\alpha)\vec{Z}^\alpha \cdot \vec{V}^\beta = \vec{V}^\beta \cdot \{ &(\nabla^\beta_{\vec{Y}^\alpha} \vec{B}^\beta)(\vec{X}^\alpha, \vec{Z}^\alpha) - \\ &-(\nabla^\beta_{\vec{X}^\alpha} \vec{B}^\beta)(\vec{Y}^\alpha, \vec{Z}^\alpha)\} - \\ &- \sum_{j \neq \alpha,\beta} Q^j(\vec{X}^\alpha, \vec{V}^\beta) \cdot B^j(\vec{Y}^\alpha, \vec{Z}^\alpha) + \\ &+ \sum_{j \neq \alpha,\beta} Q^j(\vec{Y}^\alpha, \vec{V}^\beta) \cdot B^j(\vec{X}^\alpha, \vec{Z}^\alpha) + \\ &+ \sum_{j \neq \alpha,\beta} 2\vec{Z}^\alpha \cdot Q^\alpha(\vec{A}^j(\vec{X}^\alpha, \vec{Y}^\alpha), \vec{V}^\beta) + \\ &+ 2\vec{Z}^\alpha \cdot B^\alpha(\vec{A}^\beta(\vec{X}^\alpha, \vec{Y}^\alpha), \vec{V}^\beta) \end{aligned} \tag{8}$$

$$\begin{aligned} R(\vec{X}^\alpha, \vec{Y}^\beta)\vec{Z}^\alpha \cdot \vec{V}^\beta = (\vec{Z}^\alpha \cdot (\nabla^\alpha_{\vec{X}^\alpha} \vec{B}^\alpha) + \\ +(\vec{X}^\alpha \cdot \vec{B}^\alpha, \vec{Z}^\alpha \cdot \vec{B}^\alpha))(\vec{Y}^\beta, \vec{V}^\beta) + \\ +(\vec{V}^\beta \cdot (\nabla^\beta_{\vec{Y}^\beta} \vec{B}^\beta) + (\vec{Y}^\beta \cdot \vec{B}^\beta, \vec{V}^\beta \cdot \vec{B}^\beta))(\vec{X}^\alpha, \vec{Z}^\alpha) + \end{aligned}$$

$$+\sum_{\gamma\neq\alpha,\beta}\{B^{\gamma}(\vec{X}^{\alpha},\vec{Z}^{\alpha})\cdot B^{\gamma}(\vec{Y}^{\beta},\vec{V}^{\beta})- -Q^{\gamma}(\vec{X}^{\alpha},\vec{V}^{\beta})\cdot Q^{\gamma}(\vec{Y}^{\beta},\vec{Z}^{\alpha})+ +\vec{V}^{\beta}\cdot Q^{\beta}(\vec{\Lambda}^{\gamma}(\vec{X}^{\alpha},\vec{Y}^{\beta}),\vec{Z}^{\alpha})\} \quad (9)$$

$$R(\vec{X}^{\alpha},\vec{Y}^{\beta})\vec{Z}^{\alpha}\cdot\vec{V}^{\delta}=\vec{V}^{\delta}\cdot\{(\nabla^{\delta}_{\vec{Y}^{\beta}}\vec{B}^{\delta})(\vec{X}^{\alpha},\vec{Z}^{\alpha})- -(\nabla^{\delta}_{\vec{X}^{\alpha}}\vec{Q}^{\delta})(\vec{Y}^{\beta},\vec{Z}^{\alpha})\}+ +\vec{Z}^{\alpha}\cdot B^{\alpha}(\vec{Y}^{\beta},\vec{Q}^{\beta}(\vec{X}^{\alpha},\vec{V}^{\delta}))- -\vec{V}^{\delta}\cdot B^{\delta}(\vec{Y}^{\beta},\vec{B}^{\beta}(\vec{X}^{\alpha},\vec{Z}^{\alpha}))- -\vec{Z}^{\alpha}\cdot B^{\alpha}(\vec{\Lambda}^{\alpha}(\vec{X}^{\alpha},\vec{Y}^{\beta}),\vec{V}^{\delta})+ +(\vec{Y}^{\beta}\cdot\vec{B}^{\beta},\vec{V}^{\delta}\cdot\vec{B}^{\delta})(\vec{X}^{\alpha},\vec{Z}^{\alpha})- -(\vec{X}^{\alpha}\cdot\vec{B}^{\alpha},\vec{V}^{\delta}\cdot\vec{Q}^{\delta})(\vec{Y}^{\beta},\vec{Z}^{\alpha})- -\sum_{\gamma\neq\alpha,\beta,\delta}\{\vec{Z}^{\alpha}\cdot Q^{\alpha}(\vec{\Lambda}^{\gamma}(\vec{X}^{\alpha},\vec{Y}^{\beta}),\vec{V}^{\delta})+ +B^{\gamma}(\vec{X}^{\alpha},\vec{Z}^{\alpha})\cdot Q^{\gamma}(\vec{Y}^{\beta},\vec{V}^{\delta})+ +Q^{\gamma}(\vec{X}^{\alpha},\vec{V}^{\delta})\cdot Q^{\gamma}(\vec{Y}^{\beta},\vec{Z}^{\alpha}\} \quad (10)$$

$$R(\vec{X}^{\alpha},\vec{Y}^{\beta})\vec{Z}^{\gamma}\cdot\vec{V}^{\delta}=\vec{V}^{\delta}\cdot\{(\nabla^{\delta}_{\vec{Y}^{\beta}}\vec{Q}^{\delta})(\vec{X}^{\alpha},\vec{Z}^{\gamma})- -(\nabla^{\delta}_{\vec{X}^{\alpha}}\vec{Q}^{\delta})(\vec{Y}^{\beta},\vec{Z}^{\gamma})\}+\vec{V}^{\delta}\{\cdot B^{\delta}(\vec{X}^{\alpha},\vec{Q}^{\alpha}(\vec{Y}^{\beta},\vec{Z}^{\gamma}))- -B^{\delta}(\vec{Y}^{\beta},\vec{Q}^{\beta}(\vec{X}^{\alpha},\vec{Z}^{\gamma}))-B^{\delta}(\vec{\Lambda}^{\gamma}(\vec{X}^{\alpha},\vec{Y}^{\beta}),\vec{Z}^{\gamma})\}+ +\vec{Z}^{\gamma}\cdot\{B^{\gamma}(\vec{Y}^{\beta},\vec{Q}^{\beta}(\vec{X}^{\alpha},\vec{V}^{\delta}))- -B^{\gamma}(\vec{X}^{\alpha},\vec{Q}^{\alpha}(\vec{Y}^{\beta},\vec{V}^{\delta}))-B^{\gamma}(\vec{\Lambda}^{\gamma}(\vec{X}^{\alpha},\vec{Y}^{\beta}),\vec{V}^{\delta})\}+ +(\vec{Y}^{\beta}\cdot\vec{B}^{\beta},\vec{V}^{\delta}\cdot\vec{Q}^{\delta})(\vec{X}^{\alpha},\vec{Z}^{\gamma})- -(\vec{X}^{\alpha}\cdot\vec{B}^{\alpha},\vec{V}^{\delta}\cdot\vec{Q}^{\delta})(\vec{Y}^{\beta},\vec{Z}^{\gamma})+ +\sum_{\gamma\neq\alpha,\beta,\gamma,\delta}\{Q^{j}(\vec{Y}^{\beta},\vec{V}^{\delta})\cdot Q^{j}(\vec{X}^{\alpha},\vec{Z}^{\gamma})- -Q^{j}(\vec{Y}^{\beta},\vec{Z}^{\gamma})\cdot Q^{j}(\vec{X}^{\alpha},\vec{V}^{\delta}+ +\vec{V}^{\delta}\cdot Q^{\delta}(\vec{\Lambda}^{j}(\vec{X}^{\alpha},\vec{Y}^{\beta}),\vec{Z}^{\gamma})\} \quad (11)$$

Here we use the following definitions:

$$\Lambda^{j}(\vec{X}^{\alpha},\vec{Y}^{\beta})\equiv H^{j}([\vec{X}^{\alpha},\vec{Y}^{\beta}]) \\ (\vec{Y}^{\beta}\cdot\vec{B}^{\beta},\vec{V}^{\delta}\cdot\vec{Q}^{\delta})(\vec{X}^{\alpha},\vec{Z}^{\gamma})\equiv \\ \equiv g^{ab\alpha}[\vec{V}^{\delta}\cdot Q^{\delta}(\vec{Z}^{\gamma},\vec{e}^{\alpha}_{a})\vec{X}^{\beta})\cdot B^{\beta}(\vec{X}^{\alpha},\vec{e}^{\alpha}_{b})] \\ (\nabla^{\alpha}_{\vec{X}^{\beta}}\vec{Q}^{\alpha})(\vec{Y}^{\gamma},\vec{V}^{\delta})\equiv\nabla^{\alpha}_{\vec{X}^{\beta}}\vec{Q}^{\alpha}(\vec{Y}^{\gamma},\vec{V}^{\delta})- \\ -\vec{Q}^{\alpha}(\nabla^{\gamma}_{\vec{X}^{\beta}}\vec{Y}^{\gamma},\vec{V}^{\delta})-\vec{Q}^{\alpha}(\vec{Y}^{\alpha},\nabla^{\delta}_{\vec{X}^{\beta}}\vec{V}^{\delta}) \quad (12)$$

Objects S and A are the symmetric and antisymmetric parts of B respectively. All popular splitting methods (such as the monad method Antonov V.I., et.al., 1978, the Newman-Penrose's formalism Newman E.T. and Penrose R., 1962, the Geroch's formalism Geroch R., 1971) and many others can be obtained as a special cases of this approach when we choose concrete basis of splitting.

For example, the Newman-Penrose's method corresponds to the following projection operators:

$$H'=E_{a}\otimes\theta^{a},H''=I-E_{a}\otimes\theta^{a}, \\ g=g'+g'' \quad (13)$$

$$g'=2(\theta^{\{1}\otimes\theta^{2\}}+\theta^{\{3}\otimes\theta^{3\}})+... \\ +\theta^{\{n-1}\otimes\theta^{n\}}) \quad (14)$$

The basis of one-forms $\{\theta^{a}\}, a=1,2,...n$ is a complex one in general case.

References

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